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Hyperbolic systems of conservation laws with Lipschitz continuous flux-functions: the Riemann problem

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— Dedicated to Constantine Dafermos on his 60th birthday

Abstract. For strictly hyperbolic systems of conservation laws with Lipschitz continuous flux-functions we generalize Lax's genuine nonlinearity condition and shock admissibility inequalities and we solve the Riemann problem when the left- and right-hand initial data are sufficiently close. Our approach is based on the concept of multivalued representatives of L^∞ functions and a generalized calculus for Lipschitz continuous mappings. Several interesting features arising with Lipschitz continuous flux-functions come to light from our analysis.

Keywords: hyperbolic conservation law, entropy solution, Riemann problem, Lipschitz continuous flux, multivalued representative.

Mathematical subject classification: Primary: 35L65; Secondary: 65M12.

1 Introduction

The mathematical modeling of many problems in fluid dynamics and material science often leads to nonlinear hyperbolic systems of conservation laws. Such systems consist of nonlinear partial differential equations supplemented with constitutive relations describing the behavior of the specific medium under consideration. The "flux" of each conservation law is expressed in term of the

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"conservative" variables. Quite often in the applications, the constitutive relations have different forms in different ranges of values of the conservative variables. Typical examples are found in the modeling of multi-phase flows and of elasto-plastic materials. A solid material, for instance, may have a different behavior when its density exceeds some critical value. On the other hand, the constitutive relations must often be determined by experiments. In turn, the hyperbolic systems of interest in the applications admit flux-functions which are solely Lipschitz continuous and lack the differentiability property which is customarily assumed in the mathematical theory of conservation laws.

Our general objective is to identify new features arising in discontinuous solutions of systems of conservation laws with Lipschitz continuous flux. In the present paper, we will focus attention on the so-called Riemann problem (Lax [5]) for the strictly hyperbolic system

$$u_t + f(u)_x = 0, \quad u(x,t) \in \mathcal{U},\ x \in \mathbb{R},\ t > 0, \tag{1.1}$$

supplemented with the piecewise constant initial condition

$$u(x,0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \tag{1.2}$$

We assume that the data u_l, u_r belong to $\mathcal{U} := \mathcal{B}(u_*, \delta) \subset \mathbb{R}^N$, the ball with center u_* and (small) radius δ. The function $f : \mathcal{U} \to \mathbb{R}^N$ is assumed to be Lipschitz continuous and the matrix Df to be strictly hyperbolic. Each characteristic field of Df will be assumed to be genuinely nonlinear. (Since the flux is not smooth, these notions have to be reconsidered; see the begining of Section 4 below.)

Discontinuous solutions of (1.1) satisfying an entropy condition (required for uniqueness) will be sought. Recall that the Riemann problem plays a fundamental role within the theory of conservation laws and yields many interesting informations on general solutions of (1.1). It is the basis to develop a large class of numerical schemes (Godunov scheme, random choice method, front tracking algorithm,...). Assuming that f be of class C^2 at least and δ be sufficiently small, Lax [5] constructed the entropy solution of the Riemann problem (1.1). To extend Lax's theory to Lipschitz continuous f, the difficulty is to handle possibly *discontinuous wave speeds.* We will rely here on the generalized calculus for Lipschitz continuous mappings, for which we refer to Clarke [1]. A generalized derivative is a *set of vectors* rather than a single value. We will also rely on the (related) theory developed earlier by Fillipov [4] for ordinary differential equations with discontinuous coefficients.

An outline of the content of this paper follows. A brief review of Clarke's generalized calculus is presented in Section 2. Section 3 deals with the case of scalar conservation laws, which is particularly straightforward but nevertheless of particular interest, as it allows us to exhibit the new qualitative behavior of shock waves and rarefaction waves associated with discontinuous wave speeds. Section 4 contains a general existence theory for the Riemann problem (1.1) and (1.2) for systems. Solutions satisfy a suitable generalization of Lax shock admissibility inequalities. Observe that the Riemann solution may be non-unique when the flux is not smooth, even when entropy inequalities are imposed. Finally, in Section 5, we investigate a specific example arising in fluid dynamics. A study of the Cauchy problem for systems of conservation laws with Lipschitz continuous flux-functions is in progress.

2 Generalized gradients

Let us recall here the notion of generalized gradients for Lipschitz continuous mappings and some fundamental results we will need. We follow closely the presentation in Clarke [1].

The ball in $\mathbb{R}^N$ with center u and radius r is denoted by $\mathcal{B}_N(u, r)$. By definition, given an open subset $\mathcal{U} \subset \mathbb{R}^N$, a vector-valued mapping

$$f : \mathcal{U} \to \mathbb{R}^M, \quad f(u) = (f^1(u), f^2(u), ..., f^M(u))$$

is *k-Lipschitz continuous on the set* $\mathcal{U}$ if

$$|f(u) - f(u')| \leq k\,|u - u'|, \quad u, u' \in \mathcal{U}. \tag{2.1}$$

It is *k-Lipschitz continuous near some point u* if, for some small $\epsilon > 0$ such that the ball $\mathcal{B}_N(u, \epsilon)$ is contained in $\mathcal{U}$, the function f is k-Lipschitz continuous on $\mathcal{B}_N(u, \epsilon)$. On the other hand, when f is Lipschitz continuous near some point u, by Rademacher's theorem it is differentiable almost eveywhere (for the Lebesgue measure) on any neighborhood of u on which f is Lipschitz continuous. We will denote by Ω_f the set of all the points at which f fails to be differentiable. The notation $Df(v)$ will stand for the usual $M \times N$ matrix of partial derivatives which is well-defined whenever v is a point at which the partial derivatives exist. We are led to the following definition.

Definition 2.1. The *generalized Jacobian* $\partial f(u)$ of f at the point u is the convex hull of all $M \times N$ matrices Z obtained as limits of sequences of the form $Df(u_i)$, where $u_i \to u$ and $u_i \notin \Omega_f$. In other words, we set

$$\partial f(u) := \mathrm{co}\{\lim Df(u_i) \,/\, u_i \to u,\ u_i \notin \Omega_f\}, \tag{2.2}$$

where the notation "co" stands for the convex hull of a set.

When $M = 1$, given a real-valued function $f : \mathcal{U} \to \mathbb{R}$ which is Lipschitz continuous near some point $u \in \mathbb{R}^N$, the *generalized directional derivative* of f at u in the direction $v \in \mathbb{R}^N$ is denoted by $f^\circ(u; v)$ and defined by

$$f^\circ(u; v) := \limsup_{\substack{u' \to u, \\ t \to 0+}} \frac{f(u' + t\,v) - f(u')}{t}. \tag{2.3}$$

The *generalized gradient* of f at u is denoted by $\partial f(u)$ and defined by

$$\partial f(u) := \{w \in \mathbb{R}^N \,/\, f^\circ(u; v) \geq w \cdot v \quad \text{for all } v \in \mathbb{R}^N\}. \tag{2.4}$$

Some fundamental properties of generalized gradients are summarized below.

Proposition 2.2 [1, Prop. 2.6.2]. *Let $f(u) = (f^1(u), f^2(u), \ldots, f^M(u))$ be a mapping which is Lipschitz continuous near some point $u \in \mathbb{R}^N$. Then the following statements hold:*

(a) *$\partial f(u)$ is a non-empty convex compact subset of $\mathbb{R}^{M\times N}$.*

(b) *$\partial f(u)$ is closed at u, that is, if $u_i \to u$, $Z_i \in \partial f(u_i)$, $Z_i \to Z$, then $Z \in \partial f(u)$.*

(c) *$\partial f(u)$ is upper semi-continuous at u, that is, for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $v \in \mathcal{B}_N(u, \delta)$*

$$\partial f(v) \subset \partial f(u) + \epsilon\, \mathcal{B}_{M\times N},$$

where $\mathcal{B}_{M\times N}$ is the unit ball with center 0 in the space of $M \times N$-matrices.

(d) *If each component f^i is k_i-Lipschitz continuous at u, then f is k-Lipschitz continuous at u for some constant k, and $\partial f(u) \subset k\overline{\mathcal{B}}_{M\times N}$, where $\overline{\mathcal{B}}_{M\times N}$ is the closure of $\mathcal{B}_{M\times N}$.*

(e) *$\partial f(u) \subset \partial f^1(u) \times \partial f^2(u) \times \ldots \times \partial f^M(u)$, where the latter denotes the set of all matrices whose i-th row belongs to $\partial f^i(u)$ for each i. If $M = 1$, then $\partial f(u) = \partial f^1(u)$ (i.e., the generalized gradient and the generalized Jacobian coincide).*

In general, the generalized gradient is *not* lower semi-continuous. Recall that a set-valued function g with domain $\Omega \subset \mathbb{R}^N$ and taking values in $\mathbb{R}^M$ is said to be lower semi-continuous at a point $u \in \Omega$ if, for any open subset $\mathcal{U} \subset \Omega$ such that $\mathcal{U} \cap g(u) \neq \emptyset$, there exists $\eta > 0$ such that

$$g(v) \cap \mathcal{U} \neq \emptyset, \qquad v \in \mathcal{B}_N(u, \eta).$$

To illustrate our claim, consider the real-valued function $h : \mathbb{R} \to \mathbb{R}$, $u \mapsto h(u) = |u|$. A simple calculation shows that

$$\partial h(u) = \begin{cases} \{-1\}, & u < 0, \\ [-1, 1], & u = 0, \\ \{1\}, & u > 0, \end{cases}$$

so that the generalized gradient ∂h is not lower semi-continuous at $u = 0$.

We now state some key results of the theory of Lipschitz continuous mappings, extending classical theorems which are well-known for smooth mappings.

Theorem 2.3 (Mean value theorem) [1, Prop. 2.6.5]. *Let $f : \mathcal{U} \to \mathbb{R}^M$ be Lipschitz continuous on an open convex set $\mathcal{U} \subset \mathbb{R}^N$, and let u and v some points in $\mathcal{U}$. Then, there exists a matrix $A(u, v) \in co\, \partial f([u, v])$ (where $[u, v]$ stands for the straightline segment connecting u and v) such that*

$$f(v) - f(u) = A(u, v)\,(v - u). \tag{2.5}$$

Theorem 2.4 (Chain rule formula) [1, Cor. 2.6.6]. *Let $f : \mathbb{R}^N \to \mathbb{R}^M$ be Lipschitz near u and let $g : \mathbb{R}^M \to \mathbb{R}^K$ be Lipschitz continuous near the point $f(u)$. Then, for any $v \in \mathbb{R}^N$ one has*

$$\partial(g \circ f)(u)v \subset co\,(\partial g(f(u))(\partial f(u)v)). \tag{2.6}$$

If g is continuously differentiable near $f(u)$, then equality holds (and taking the convex hull is superfluous).

Theorem 2.5 (Inverse mapping theorem) [1, Th. 7.1.1]. *Let f be Lipschitz continuous near a given point $u_0 \in \mathbb{R}^N$. If $\partial f(u_0)$ is non-singular, in the sense that every matrix of the generalized Jacobian $\partial f(u_0)$ is non-singular, then there exist neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of u_0 and $f(u_0)$, respectively, and a unique Lipschitz function $g : \mathcal{V} \to \mathbb{R}^N$ such that*

$$g(f(u)) = u \quad \textit{for every } u \in \mathcal{U}$$

and

$$f(g(v)) = v \quad \textit{for every } v \in \mathcal{V}.$$

We will also need the implicit function theorem. Consider a mapping $h : \mathbb{R}^M \times \mathbb{R}^K \to \mathbb{R}^K$, together with the implicit equation

$$h(v, w) = 0 \quad \text{where } (v, w) \in \mathbb{R}^M \times \mathbb{R}^K. \tag{2.7}$$

Assume that h is Lipschitz continous near the point $(v_0, w_0) \in \mathbb{R}^M \times \mathbb{R}^K$, and that (v_0, w_0) satisfies the equation (2.7). Denote $\pi_w \partial h(v_0, w_0)$ the projection in the w-direction, that is, the set of all $K \times K$ matrices A such that, for some $K \times M$ matrix B, the $K \times (K + M)$ matrix $\big(B\ A\big)$ belongs to $\partial h(v_0, w_0)$.

Theorem 2.6 (Implicit mapping theorem) [1, Cor. 7.1.1]. *Under the above notation and assumptions, suppose that each matrix of the set $\pi_w \partial h(v_0, w_0)$ is of maximal rank. Then, there exists a neighborhood $\mathcal{V}$ of v_0 and a unique Lipschitz continuous function $r : \mathcal{V} \to \mathbb{R}^K$ such that $r(v_0) = w_0$ and*

$$h(v, r(v)) = 0 \quad \textit{for every } v \in \mathcal{V}. \tag{2.8}$$

3 Scalar conservation laws

To begin with, in this section we consider the equation (1.1) when $N = 1$ and investigate the Riemann problem. Recall that we solely assume that the flux f belongs to $W^{1,\infty}(\mathbb{R})$. For such a function of a single variable one can set

$$\begin{aligned} f'_+(u) &= \limsup_{\substack{v \to u \\ h \to 0+}} \frac{f(v+h) - f(v)}{h}, \\ f'_-(u) &= \liminf_{\substack{v \to u \\ h \to 0+}} \frac{f(v+h) - f(v)}{h}. \end{aligned} \tag{3.1}$$

Proposition 3.1. *At every point $u \in \mathbb{R}$ we have*

$$\partial f(u) = [f'_-(u), f'_+(u)]. \tag{3.2}$$

Proof. First of all by the definition (2.3) we have

$$f'_+(u) = f^\circ(u; 1)$$

and

$$\begin{aligned} f'_-(u) &= -\limsup_{\substack{v\to u \\ h\to 0+}} \left(-\frac{f(v+h)-f(v)}{h} \right) \\ &= -\limsup_{\substack{v\to u \\ h\to 0+}} \frac{(-f)(v+h)-(-f)(v)}{h} \\ &= -(-f)^\circ(u;1) = -f^\circ(u;-1). \end{aligned} \tag{3.3}$$

By definition, $w \in \partial f(u)$ if and only if

$$w \cdot v \le f^\circ(u; v), \qquad v \in \mathbb{R}.$$

Since both sides of the last inequality are positively homogeneous of degree one, the condition reduces to

$$w \le f^\circ(u;1) \text{ and } -w \le f^\circ(u;-1).$$

From (3.3) we also easily deduce that

$$\begin{aligned} w &\le f^\circ(u;1) = f'_+(u), \\ w &\ge -f^\circ(u;-1) = f'_-(u), \end{aligned}$$

which completes the proof. □

The wave speed

$$\lambda(u) := f'(u)$$

solely belongs to $L^\infty(\mathbb{R})$. The associated shock speed defined by

$$\sigma(u,v) = \frac{f(v)-f(u)}{v-u} \tag{3.4}$$

is a Lipschitz function of its argument away from the diagonal $\{u = v\}$. Observe that given some state u_0 and for specific sequences $u, v \to u_0$ we may reach any value within the interval $\partial f(u_0)$.

We will generalize here Oleinik's construction of the solution of the Riemann problem (1.1)-(1.2) to the case of a Lipschitz continuous flux. To begin with, we will review the notion of generalized inverse of monotone mappings. Consider a function $h : [a,b] \to \mathbb{R}$ which is non-decreasing on a closed interval $[a,b] \subset \mathbb{R}$, i.e.,

$$y_0, y_1 \in [a,b], \qquad y_0 \ge y_1 \Longrightarrow h(y_0) \ge h(y_1).$$

Then, the function h has locally bounded variation and its set of discontinuity points is at most countable. Moreover, at each discontinuity point y we can define left- and right-hand limits denoted by $h_-(y)$ and $h_+(y)$, respectively. Since h is non-decreasing, there is no ambiguity between this notation and the one in (3.1). At points of continuity we have obviously that $h_-(y) = h_+(y) = h(x)$. The functions h_- and h_+ are the left- and right-continuous representatives of the function h. For each $\xi \in [h(a), h(b)]$ consider the set

$$G(\xi) := \{y \in [a, b] \,/\, h(y) = \xi\}. \tag{3.5}$$

We can distinguish between three cases: $G(\xi)$ may be either a single point, or an interval $I \subset [a, b]$ with distinct endpoints, or the empty set. We state without proof (see [3]):

Lemma and Definition 3.2. *Let $h : [a, b] \to \mathbb{R}$ be a non-decreasing function. Its (non-decreasing)* generalized inverse *denoted by $h^{-1} : [h(a), h(b)] \to [a, b]$ is defined as follows at each $\xi \in [h(a), h(b)]$:*

(i) *If $G(\xi) = \{y\}$, then we set*

$$h^{-1}(\xi) = y.$$

(ii) *If $G(\xi)$ is an interval $I \subset [a, b]$ with distinct endpoints $y_0 < y_1$, then we can pick up any value*

$$h^{-1}(\xi) \in I,$$

for instance the lower bound y_0 of the interval I. In that case, ξ is a point of discontinuity of the function h, the set of such points ξ being of course at most countable.

(iii) *If $G(\xi) = \emptyset$, then there exists a unique value $y \in [a, b]$ such that $h_-(y) \le \xi \le h_+(y)$. Then we set*

$$h^{-1}(\xi) = y$$

and we have

$$h^{-1}(\xi) = y \quad \text{for all values} \quad \xi \in [h_-(y), h_+(y)].$$

The function $h^{-1}(\xi)$ is non-decreasing in ξ. Moreover, if h is strictly increasing, then its generalized inverse h^{-1} is continuous.

This notion is obviously consistent with the standard definition when h is invertible. Throughout the present paper, the inverse of a monotone function is always understood in the sense above.

Our main result in this section is the following one.

Theorem 3.3. *Consider a Lipschitz continuous flux-function f and some Riemann data u_l and u_r such that (for definiteness) $u_l < u_r$. Let*

$$\tilde{f} : [u_l, u_r] \to \mathbb{R}$$

be the (Lipschitz continuous) convex hull *of f on the interval $[u_l, u_r]$. Consider also the generalized inverse of $\tilde{f}'$ in the sense of Definition 3.2*

$$g := \left(\tilde{f}'\right)^{-1} : [\tilde{f}'_+(u_l), \tilde{f}'_-(u_r)] \to \mathbb{R}.$$

Then, the explicit formula

$$u(x,t) = \begin{cases} u_l, & x < t\, \tilde{f}'_+(u_l), \\ g(x/t), & t\, \tilde{f}'_+(u_l) < x < t\, \tilde{f}'_-(u_r), \\ u_r, & x > t\, \tilde{f}'_-(u_r), \end{cases} \tag{3.6}$$

defines a function with bounded variation which is the entropy solution of the Riemann problem (1.1)-(1.2) *satisfying Oleinik's entropy inequalities.*

Proof. Setting

$$v(\xi) := u(x,t), \quad \xi = x/t,$$

we must show that the Borel measure

$$\mu := -\xi \frac{dv}{d\xi} + \frac{d}{d\xi}\big(f(v)\big) = (-\xi + \hat{f}'(v))\, \frac{dv}{d\xi} \tag{3.7}$$

vanishes identically, where $dv/d\xi$ is a measure and Volpert's superposition $\hat{f}'(v)$ is the function of bounded variation defined by

$$\hat{f}'(v)(\xi) := \begin{cases} f'_-(v(\xi)) & \text{at points of continuity of } v, \\ \displaystyle\int_0^1 f'\big(\theta\, v_-(\xi)) + (1-\theta)\, v_+(\xi)\big)\, d\theta & \text{at points of jump of } v. \end{cases}$$

Here, the representative f'_- is chosen for definiteness, only. See [3] for a justification of the above chain rule. Given an arbitrary Borel set B we can introduce the decomposition

$$\mu(B) = \mu(B_c) + \sum_m \mu(\{\xi_m\}), \qquad B = B_c \cup \{\xi_1, \xi_2, ...\},$$

in which v is continuous at every point of B_c and discontinuous at each $\xi_1, \xi_2, \ldots$ We can now deal with the set of points of continuity and of points of jump separately.

First of all, suppose that f is convex on the interval $[u_l, u_r]$, so that

$$\tilde{f}(u) = f(u), \quad u \in [u_l, u_r].$$

We distinguish between two situations. If v is continuous at some point ξ and that f is differentiable at $v(\xi)$, then we have by definition

$$\hat{f}'(v(\xi)) = f'(v(\xi)).$$

Since v is precisely the inverse of f' this yields

$$f'(v(\xi)) = \xi.$$

If now v is continuous at some point ξ but f is not differentiable at $v(\xi)$, i.e.,

$$f'_-(v(\xi)) < f'_+(v(\xi)),$$

then we have

$$v(f'_-(v(\xi))) = v(f'_+(v(\xi))).$$

Since v is monotone, v remains constant on the non-trivial interval

$$[f'_-(v(\xi)), f'_+(v(\xi))]$$

(which contains ξ). We conclude that the measure $dv/d\xi$ vanishes identically in this interval. Collecting our conclusions in both cases, it follows that if B is a subset of the set of continuity points of v, then

$$\mu(B) = 0.$$

Next, let ξ be any point of discontinuity of v. We have

$$\mu\big(\{\xi\}\big) = -\xi\,(v_+(\xi) - v_-(\xi)) + f(v_+(\xi)) - f(v_-(\xi)).$$

Since f' is the inverse of v, f' must be constant on the interval $[v_-(\xi), v_+(\xi)]$, that is,

$$f'(u) = \xi, \quad u \in [v_-(\xi), v_+(\xi)].$$

Therefore, $w \mapsto f(w)$ is affine on this interval and is given by

$$f(w) = f(v_-(\xi)) + \xi\,\big(u - v_-(\xi)\big), \quad w \in [v_-(\xi), v_+(\xi)],$$

and in particular we obtain

$$\mu(\{\xi\}) = 0.$$

This completes the proof that (3.6) provides a solution of the scalar conservation law (1.1), at least when the flux f is assumed to be convex.

To treat the general case when f need not be convex let us set

$$\mathcal{A} := \{w \,/\, \tilde{f}(w) = f(w)\}.$$

Since both f and $\tilde{f}$ are continuous, the set $\mathcal{A}$ is closed and can be decomposed in a countable union of closed intervals, say $[a_n, b_n]$, $n = 1, 2, \cdots$. In each interval $[a_n, b_n]$ the function f is convex and our arguments in the first part of this proof show immediately that the formula (3.6) determine a weak solution of (1.1) if the initial data lie in $[a_n, b_n]$. The remaining set $\mathcal{A}^c$ is open and, therefore, can be decomposed into a countable union of open intervals (c_n, d_n), $n = 1, 2, \cdots$. Without loss of generality we can assume that $c_n, d_n \in \mathcal{A}$, so that

$$\tilde{f}'_-(c_n) = f'_-(c_n) \quad \text{and} \quad \tilde{f}'_+(d_n) = f'_+(d_n).$$

By definition, $\tilde{f}$ must be affine on the interval $[c_n, d_n]$. Thus, we get

$$\tilde{f}'_-(c_n) = f'_-(c_n) = \tilde{f}'_+(d_n) = f'_+(d_n) =: \lambda. \tag{3.8}$$

The conditions (3.8) imply that, at the point λ, the function v has a jump discontinuity and

$$v_-(\lambda) = c_n \quad \text{and} \quad v_+(\lambda) = d_n.$$

Then we have

$$\mu(\{\lambda\}) = -\lambda\left(v_+(\lambda) - v_-(\lambda)\right) + f(v_+(\lambda)) - f(v_-(\lambda)) = 0.$$

Therefore, if the initial data belong to the interval $[c_n, d_n]$, then λ is the unique point of discontinuity of v, and for $\xi \neq \lambda$, the function v is constant. This means that the function v (or, more precisely, $u = u(x, t)$) has a discontinuity propagating at the speed λ.

Finally, if the initial data take values in several distinct intervals, we can find a decomposition the formula (3.6) to reduce the problem to solutions with data belonging to a single interval.

To complete the proof, it remains to check that Oleinik's entropy inequalities hold at each discontinuity connecting some left-hand state u_- to a right-hand state u_+, that is,

$$\sigma(u_-, u_+) \le \sigma(u_-, w), \qquad w \in (u_-, u_+). \tag{3.9}$$

Consider the shock wave determined earlier from the conditions (3.8), with now

$$u_- = c_n, \quad u_+ = d_n, \quad \sigma(u_-, u_+) = \lambda.$$

Since $\tilde{f}$ is the convex hull of f and is distinct from f at each point of the interval (u_-, u_+), we have

$$\tilde{f}(w) < f(w), \quad w \in (u_-, u_+). \tag{3.10}$$

Thus, (3.10) yields for all $w \in (u_-, u_+)$

$$\begin{aligned}\sigma(u_-, w) &= \frac{f(w) - f(u_-)}{w - u_-} > \frac{\tilde{f}(w) - f(u_-)}{w - u_-} \\ &= \frac{\tilde{f}(w) - \tilde{f}(u_-)}{w - u_-} = \lambda = \sigma(u_-, u_+).\end{aligned}$$

The proof of Theorem 3.3 is complete. □

To illustrate some interesting features of the loss of regularity in the flux-function f, let us discuss an example. Suppose that, for some critical value $u_* \in \mathbb{R}$, the flux f is a smooth convex function in both intervals $u < u_*$ and $u > u_*$, but the speed $\lambda(u) = f'(u)$ is discontinuous at u_* with

$$\lambda_-(u_*) < \lambda_+(u_*),$$

so that the flux f is globally convex but solely Lipschitz continuous. Then, on one hand, a rarefaction wave connecting $u_l < u_*$ to $u_r > u_*$ contains a *constant state:*

$$u(x,t) = \begin{cases} u_l, & x < t\,\lambda(u_l), \\ f'^{-1}(x/t), & t\,\lambda(u_l) < x < t\,\lambda_-(u_*), \\ u_*, & t\,\lambda_-(u_*) < x < t\,\lambda_+(u_*), \\ f'^{-1}(x/t), & t\,\lambda_+(u_*) < x < t\,\lambda(u_r), \\ u_r, & x > t\,\lambda(u_r). \end{cases}$$

On the other hand, concerning shock waves, it is easy to see that the shock speed always has a limiting value if one data coincides with u_* while the other approaches u_*, namely

$$\sigma(u_*, u_r) \to \lambda_-(u_*), \quad u_r \to u_*$$

and

$$\sigma(u_l, u_*) \to \lambda_+(u_*), \quad u_l \to u_*.$$

However, the speed $\sigma(u_l, u_r)$ has no limit when both $u_l, u_r \to u_*$ and instead we obtain

$$\liminf_{u_l, u_r \to u_*} \sigma(u_l, u_r) = \lambda_-(u_*)$$

and

$$\limsup_{u_l, u_r \to u_*} \sigma(u_l, u_r) = \lambda_+(u_*).$$

4 Riemann problem for Systems

We now turn to general $N \times N$ systems (1.1) with Lipschitz continuous flux f and, following Lax's approach [5], we construct explicitly the entropy solution of the Riemann problem. As is usual, we restrict attention to self-similar solutions, $u(x, t) = u(y)$ with $y = x/t$ and rely on two fundamental families of solutions, the shock waves and the rarefaction waves.

Let us first introduce a *notion of strict hyperbolicity* for systems of conservation laws with non-smooth flux. Recall that all of the values u under consideration will remain in a ball $\mathcal{U} := \mathcal{B}(u_*, \delta_0)$ with sufficiently small radius δ_0. The system (1.1) is assumed to be strictly hyperbolic. We fix some $N \times N$ matrix A^* with real and distinct eigenvalues

$$\lambda_1^* < \ldots < \lambda_N^*$$

and corresponding basis of left- and right-eigenvectors l_j^* and r_j^*, $j = 1, \ldots, N$, respectively. After normalization we can have $|r_i^*| = 1$, $l_i^* \cdot r_j^* = 0$ if $i \neq j$ and $l_j^* \cdot r_j^* = 1$. We assume that the Jacobian matrix of the flux $f : \mathcal{U} \mapsto \mathbb{R}^N$ remains close to A^*, i.e.,

$$\|Df(u) - A^*\| \leq \eta \qquad \text{for almost every } u \in \mathcal{B}(u_*, \delta_0), \tag{4.1}$$

where the constants δ_0 and $\eta > 0$ are sufficiently small and $\|B\|$ denotes the Euclidian norm of a matrix B. For η small enough, (4.1) implies that, for almost every u in $\mathcal{B}(u_*, \delta_0)$, the matrix $Df(u)$ has N real and distinct eigenvalues

$$\lambda_1(u) < \ldots < \lambda_N(u)$$

and corresponding basis of left- and right-eigenvectors $l_j(u)$, $r_j(u)$, $j = 1, \ldots, N$, respectively. Moreover, for some uniform constant $C > 0$, (4.1) also implies for $j = 1, \ldots, N$ and for almost every $u \in \mathcal{B}(u_*, \delta_0)$

$$\begin{aligned} |\lambda_j(u) - \lambda_j^*| &\leq C\,\eta, \\ |l_j(u) - l_j^*| &\leq C\,\eta, \\ |r_j(u) - r_j^*| &\leq C\,\eta. \end{aligned} \tag{4.2}$$

Thanks to the definition of generalized Jacobian (see (2.4) in Section 2) and the property of convex hulls, the properties in (4.2) remain valid for the generalized Jacobian $\partial f(u)$, that is,

$$\|\bar{A} - A^*\| \le \eta \quad \text{for all } \bar{A} \in \partial f(u),\ u \in \mathcal{B}(u_*, \delta_0). \tag{4.3}$$

Let $\Lambda_j(u)$ be the set of all j-eigenvalues of the matrices belonging to the set $\partial f(u)$. In view of (4.3), for each $\bar{\lambda}_j \in \Lambda_j(u)$ there exists a left-eigenvector $\bar{l}_j$ and a right-eigenvector $\bar{r}_j$ such that

$$\begin{aligned} |\bar{\lambda}_j - \lambda_j^*| &\le C\,\eta, \\ |\bar{l}_j - l_j^*| &\le C\,\eta, \\ |\bar{r}_j - r_j^*| &\le C\,\eta. \end{aligned} \tag{4.4}$$

The corresponding sets of "normalized" left- and right-eigenvectors will be denoted by $L_j(u)$ and $R_j(u)$, $j = 1, \dots, N$, respectively:

$$\begin{aligned} |\bar{l}_j - l_j^*| \le C\,\eta \quad &\text{for all } \bar{l}_j \in L_j(u), \\ |\bar{r}_j - r_j^*| \le C\,\eta \quad &\text{for all } \bar{r}_j \in R_j(u). \end{aligned}$$

For $u \neq v$ we denote by $\Lambda_j(u, v)$ the set of j-eigenvalues $\bar{\lambda}_j$ of matrices $A(u, v) \in \mathrm{co}\,(\partial f([u, v]))$ satisfying

$$A(u, v)\,\big(v - u\big) = f(v) - f(v).$$

Second, we state a *generalized notion of genuine nonlinearity* for Lipschitz continuous flux-functions. Basically, we impose that characteristic speeds and wave speeds are monotone along wave curves. Precisely, for each $j = 1, \dots, N$ each Lipschitz continuous curve $(-\epsilon_0, \epsilon_0) \ni \epsilon \mapsto v(\epsilon) \in \mathcal{U}$ satisfying

$$|v'(\epsilon) - r_j^*| \le C\,\eta \quad \text{for almost every } \epsilon \in (-\epsilon_0, \epsilon_0), \tag{4.5a}$$

and each measurable selections $(-\epsilon_0, \epsilon_0) \ni \epsilon \mapsto \lambda(\epsilon), \sigma(\epsilon) \in \mathbb{R}$ satisfying

$$\sigma(\epsilon) \in \Lambda_j(v(0), v(\epsilon)), \quad \lambda(\epsilon) \in \Lambda_j(v(\epsilon)), \tag{4.5b}$$

the functions $\lambda(\epsilon)$ and $\sigma(\epsilon)$ are *(strictly) increasing*. Moreover, for some uniform constant $m > 0$ and all $-\epsilon_0 < \epsilon_1 < \epsilon_2 < \epsilon_0$, we have

$$\lambda(\epsilon_2) - \lambda(\epsilon_1) \ge m(\epsilon_2 - \epsilon_1). \tag{4.6}$$

This assumption represents a direct generalization of Lax's concept.

Finally, we assume the following regularity assumption on the flux along wave curves: for each Lipschitz continuous curve v satisfying (4.5), the function f is continuously differentiable at $v(\epsilon)$ for almost every $\epsilon \in (-\epsilon_0, \epsilon_0)$. For example, we will use later (when dealing with rarefaction waves) that the following chain rule holds

$$f(v(\epsilon))' = Df(v(\epsilon))\, v(\epsilon)' \quad \text{for almost every } \epsilon \in (-\epsilon_0, \epsilon_0).$$

We begin with the derivation of two classes of elementary solutions, which will be used next to solve the Riemann problem. A *shock wave* traveling at the speed σ

$$u(x,t) = \begin{cases} u_0, & x < \sigma\, t, \\ u, & x > \sigma\, t, \end{cases}$$

with $u_0, u \in \mathcal{U}$, must satisfy the Rankine-Hugoniot relations:

$$-\sigma\,(u - u_0) + f(u) - f(u_0) = 0. \tag{4.7}$$

The Hugoniot set of all states u connected to a fixed state u_0 decomposes into N curves, which must be firther constrained with an entropy condition. Observe that, because the flux f is solely Lipschitz continuous, wave speeds are not defined as functions but rather as subsets of $\mathbb{R}$. Accordingly, we need a generalization of Lax shock admissibility inequalities, stated in (4.8) below.

Theorem 4.1. *Assume that the system* (1.1) *is strictly hyperbolic and genuinely nonlinear. For each $i = 1, \ldots, N$, there exist $\delta_1 < \delta_0, \epsilon_1 > 0$, and a unique Lipschitz continuous mapping*

$$\varphi_i : (-\epsilon_1, 0] \times \mathcal{B}(u_*, \delta_1) \to \mathcal{B}(u_*, \delta_0),$$

and a unique bounded measurable mapping

$$\sigma_i : (-\epsilon_1, 0] \times \mathcal{B}(u_*, \delta_1) \to \mathbb{R}\,,$$

which is locally Lipschitz continuous on $(-\epsilon_1, 0) \times \mathcal{B}(u_*, \delta_1)$, such that the following holds.

For every $\epsilon \in (-\epsilon_1, 0)$ and $u_0 \in \mathcal{B}(u_*, \delta_1)$ the left-hand state u_0 can be connected to the right-hand state $u := \varphi_i(\epsilon; u_0)$ by an i-shock wave with speed

$\varphi_i(\epsilon_1; u_0)$. *That is, Rankine-Hugoniot relations* (4.7) *hold together with the following* generalized Lax shock admissibility inequalities

$$\Lambda_i(u_0) \ni \sigma_i(0; u_0) > \sigma_i(\epsilon; u_0) > \sigma_i\big(\epsilon; \varphi_i(\epsilon; u_0)\big) \in \Lambda_i\big(\varphi_i(\epsilon; u_0)\big). \tag{4.8}$$

The functions σ_i is increasing with respect to ϵ and

$$\begin{aligned}
&\varphi_i(0; u_0) = u_0, \\
&\partial\varphi_i(0; u_0) \subset R_i(u_0), \\
&\sigma_i(0; u_0) \in \Lambda_i(u_0).
\end{aligned} \tag{4.9}$$

Note in passing that the following Taylor-like expansion follows from Theorem 4.1

$$\varphi_i(\epsilon; u_0) \in u_0 + \epsilon\, R_i(u_0) + o(\epsilon)\, \mathcal{B}(0, 1), \tag{4.10}$$

which determines the local behavior of the shock curve.

Proof. By the (generalized) mean-value theorem stated in Theorem 2.3, there exists a matrix-valued and measurable function $A(u_0, u) \in \operatorname{co}\,(\partial f([u_0, u]))$ such that

$$f(u) - f(u_0) = A(u_0, u)\,(u - u_0). \tag{4.11}$$

Hence, the Rankine-Hugoniot relations (4.7) become

$$\big(-\sigma\, I + A(u_0, u)\big)\,(u - u_0) = 0, \tag{4.12}$$

where I denotes the identity matrix.

Let us fix u_0. Thanks to (4.3), the averaging matrix $A(u_0, u)$ satisfies

$$\|A(u_0, u) - A^*\| \le \eta. \tag{4.13}$$

Let $\lambda_i(u_0, u)$ and $r_i(u_0, u)$, $i = 1, \ldots, N$ be the eigenvalues and right-eigenvectors of $A(u_0, u)$, respectively. The equations (4.12) take the following equivalent form: There exists $i = 1, \ldots, N$ and a real α such that

$$u - u_0 = \alpha\, r_i(u_0, u), \quad \sigma = \lambda_i(u_0, u). \tag{4.14}$$

The main difficulty in order to solve (4.14) lies in the lack of regularity of the eigenvectors and eigenvalues of $A(u_0, u)$.

Consider (4.7) and multiply it successively by each left-eigenvector l_j^*:

$$-\sigma(u)\, l_j^* \cdot (u - u_0) + l_j^* \cdot \big(f(u) - f(u_0)\big) = 0, \quad j = 1, \ldots, N. \tag{4.15}$$

Fix some index i. The i-th equation in (4.15) determines the shock speed:

$$\sigma(u) = \frac{l_i^* \cdot \big(f(u) - f(u_0)\big)}{l_i^* \cdot (u - u_0)} = \frac{l_i^* \cdot A(u_0, u)\,(u - u_0)}{l_i^* \cdot (u - u_0)}. \tag{4.16}$$

We are going to show that there exists a curve $\epsilon \mapsto \varphi_i(\epsilon; u_0)$ defined for small $|\epsilon|$ such that along this curve, the shock speed

$$\sigma_i(\epsilon; u_0) := \sigma(\varphi_i(\epsilon; u_0))$$

determined by (4.16) fulfills the system of N equations (4.15).

The formula (4.16) requires u to satisfy $l_i^* \cdot (u - u_0) \neq 0$. For that reason, we restrict attention to the cone

$$C_{\gamma,i}(u_0) := \big\{u \in \mathcal{U} \,/\, |l_i^* \cdot (u - u_0)| > \gamma\, |u - u_0|\big\},$$

where $\gamma \in [|l_i^*| \cdot \alpha, |l_i^*|)$ is a fixed constant, for some $\alpha \in (0, 1)$. Note that u_0 does not belong to this open cone. Note also that the Lipschitz regularity of the shock speed, as stated in the theorem, follows immediately.

Then, observe that the shock speed remains uniformly bounded in the cone $C_{\gamma,i}(u_0)$, namely

$$\begin{aligned}
\sigma(u) &= \frac{l_i^* \cdot A^*\,(u - u_0)}{l_i^* \cdot (u - u_0)} + \frac{l_i^* \cdot \big(A(u_0, u) - A^*\big)\,(u - u_0)}{l_i^* \cdot (u - u_0)} \\
&= \lambda_i^* + \frac{l_i^* \cdot \big(A(u_0, u) - A^*\big)\,(u - u_0)}{l_i^* \cdot (u - u_0)}.
\end{aligned}$$

In particular, we find

$$|\sigma(u) - \lambda_i^*| \leq \frac{|l_i^*|}{\gamma}\, \|A(u_0, u) - A^*\| \leq C\,\eta. \tag{4.17}$$

On the other hand, the shock speed is continuous on $C_{\gamma,i}(u_0)$. However, in general, it cannot be extended by continuity to $u = u_0$.

Plugging the expression (4.16) of the shock speed in the relations (4.15) yields for $j \neq i$:

$$\begin{aligned}
F_j(u) := &- \frac{l_i^* \cdot \big(f(u) - f(u_0)\big)}{l_i^* \cdot (u - u_0)}\, l_j^* \cdot (u - u_0) \\
&+ l_j^* \cdot \big(f(u) - f(u_0)\big) = 0.
\end{aligned} \tag{4.18}$$

Since f is Lipschitz continuous and the shock speed is bounded, the functions F_j are locally Lipschitz continuous on $C_{\gamma,i}(u_0)$. They are easily extended by continuity to $u = u_0$ by setting

$$F_j(u_0) = 0.$$

We now prove that the functions F_j are Lipschitz continuous up to the point u_0. To this end, it is sufficient to check that the gradients ∇F_j are uniformly bounded. We rewrite F_j in the form

$$F_j(u) = -\frac{l_j^* \cdot (u - u_0)}{l_i^* \cdot (u - u_0)}\, l_i^* \cdot \big(f(u) - f(u_0)\big) + l_j^* \cdot \big(f(u) - f(u_0)\big),$$

so that for almost every $u \in C_{\gamma,i}(u_0)$

$$\begin{aligned}
\nabla F_j(u) = &- \frac{l_i^* \cdot \big(f(u) - f(u_0)\big)}{l_i^* \cdot (u - u_0)}\, l_j^* \\
&+ \frac{l_j^* \cdot (u - u_0)}{\big(l_i^* \cdot (u - u_0)\big)^2}\, l_i^* \cdot \big(f(u) - f(u_0)\big)\, l_i^* \\
&- \frac{l_j^* \cdot (u - u_0)}{l_i^* \cdot (u - u_0)}\, l_i^* \cdot Df(u) + l_j^* \cdot Df(u).
\end{aligned} \tag{4.19}$$

Since f is Lipschitz continuous and u belongs to the cone, every term in the right-hand side of the formula above is uniformly bounded.

Our objective now is to apply the implicit function theorem to the functions F_j. We claim that the $N - 1$ vectors $\nabla F_j(u)$ are linearly independent in $\mathbb{R}^N$, uniformly for *almost every* $u \in \mathfrak{U}$. We can rewrite the expression of the gradient as:

$$\begin{aligned}
\nabla F_j(u) \ &= K_1\, l_j^* + K_2(u)\, l_j^* + K_3(u)\, l_i^* \\
&\quad + K_4(u) l_i^* \cdot \big(Df(u) - A^*\big) + l_j^* \cdot \big(Df(u) - A^*\big)
\end{aligned} \tag{4.20}$$

with

$$\begin{aligned}
K_1 &= \lambda_j^* - \lambda_i^*, \\
K_2(u) &= -\frac{l_i^* \cdot \big(A(u_0, u) - A^*\big)(u - u_0)}{l_i^* \cdot (u - u_0)}, \\
K_3(u) &= \frac{l_j^* \cdot (u - u_0)}{\big(l_i^* \cdot (u - u_0)\big)^2}\, l_i^* \cdot \big(A(u_0, u) - A^*\big)(u - u_0), \\
K_4(u) &= -\frac{l_j^* \cdot (u - u_0)}{l_i^* \cdot (u - u_0)}.
\end{aligned}$$

We estimate these coefficients successively. Observe that K_1 is a constant independent of u. Next, using (4.13) and the fact that u belongs to the cone, we get for some constant $C' > 0$

$$|K_2(u)l_j^*| \leq |K_2(u)||l_j^*| \leq \frac{1}{\gamma}\,|l_i^*||l_j^*|\,\eta \leq C'\,\eta.$$

Similarly, we obtain

$$|K_3(u)l_i^*| \leq |K_3(u)||l_i^*| \leq \left(\frac{|l_i^*|}{\gamma}\right)^2\,|l_j^*|\,\eta \leq C'\,\eta.$$

This proves that the second and third term in the right-hand side of (4.20) are of order η. The coefficient K_4 is of order 1 but, using (4.1), we have the estimate (for some constant $C' > 0$)

$$\left|K_4(u)\,l_i^* \cdot \left(Df(u) - A^*\right)\right| \leq \frac{1}{\gamma}\,|l_i^*|\,|l_j^*|\,\eta \leq C'\,\eta$$

and, thus, the fourth term in the the right-hand side of (4.20) is of order η as well. Finally, the last term satisfies

$$\left|l_j^* \cdot \left(Df(u) - A^*\right)\right| \leq C'\,\eta.$$

It follows from the above estimates that for some uniform constant C'

$$|\nabla F_j(u) - K_1\,l_j^*| \leq C'\,\eta \qquad \text{for almost every } u. \tag{4.21}$$

The functions F_j are defined within the cone only. Let $\tilde{F}_j$ be a Lipschitz continuous extension of F_j to the whole set $\mathfrak{U}$ such that (4.21) still holds for the function $\tilde{F}$:

$$|\nabla \tilde{F}_j(u) - K_1\,l_j^*| \leq C'\,\eta \qquad \text{for almost every } u.$$

Therefore, by the property of generalized gradients,

$$|\partial \tilde{F}_j(u) - K_1\,l_j^*| \leq C'\,\eta \qquad \text{for every } u \in \mathfrak{U}. \tag{4.22}$$

Since $\{l_j^*, j = 1, 2, \dots, N\}$ is a basis, we can always assume that η is small enough so that (4.22) implies that the set made of the vector l_i^* and any selection of $N-1$ vectors in $\partial \tilde{F}_j(u)$, $j \neq i$ is a basis.

Consider the function $G = G(\epsilon, w) \in \mathbb{R}^N$ defined for (ϵ, w) in a neighborhood of $(0, 0) \in \mathbb{R} \times \mathbb{R}^N$ by

$$\begin{aligned} G_i(\epsilon, w) &:= l_i^* \cdot w. \\ G_j(\epsilon, w) &:= \tilde{F}_j(u_0 + \epsilon\, r_i^* + w) \quad \text{for } j \neq i. \end{aligned}$$

Differentiating with respect to w we get, for almost every (ϵ, w),

$$\begin{aligned} \partial_w G_i(\epsilon, w) &= \{l_i^*\}, \\ \partial_w G_i(\epsilon, w) &= \partial_u \tilde{F}_j(u_0 + \epsilon r_i^* + w) \quad \text{for } j \neq i. \end{aligned}$$

Observe that

$$G(0, 0) = 0$$

and, as explained earlier,

$$\partial_w G(0, 0) \subset \partial_w G_1(0, 0) \times \partial_w G_2(0, 0) \times \cdots \times \partial_w G_N(0, 0)$$

is of maximal rank. Applying the implicit function theorem (Theorem 2.6) to the function G, we see that there exist an $\epsilon_1 > 0$ and a unique Lipschitz function $w_i(\cdot, u_0) : (-\epsilon_1, \epsilon_1) \to \mathbb{R}^N$ such that $w_i(0, u_0) = 0$ and

$$\begin{aligned} &\tilde{F}_j\big(u_0 + \epsilon\, r_i^* + w_i(\epsilon, u_0)\big) = 0 \quad \text{for } j \neq i, \\ &l_i^* \cdot w_i(\epsilon) = 0, \quad \epsilon \in (-\epsilon_1, \epsilon_1). \end{aligned} \tag{4.23}$$

Let us define

$$\begin{aligned} \varphi_i(\epsilon; u_0) &= u_0 + \epsilon\, r_i^* + w_i(\epsilon, u_0), \\ \sigma_i(\epsilon; u_0) &= \sigma(\varphi_i(\epsilon; u_0)). \end{aligned}$$

We need to show that these functions φ_i, σ_i are the ones for which we are seaching. Taking the derivative in ϵ to the equations of (4.23) and applying the chain rule formula (2.6), we have

$$\begin{aligned} 0 &= l_i^* \cdot w_i'(\epsilon), \\ 0 &= A_j \cdot (r_i^* + w_i'(\epsilon, u_0)), \quad \text{for a.e. } \epsilon \in (-\epsilon_1, \epsilon_1), \quad j \neq i, \end{aligned}$$

for some $A_j \in \partial \tilde{F}_j(u_0 + \epsilon r_j^* + w_i(\epsilon, u_0))$. Observe that the vector A_j is closed to $K_1 l_j^*$ in the sense that $\partial \tilde{F}_j(u_0 + \epsilon r_i^* + w_i(\epsilon, u_0))$ fulfills the estimate (4.22). By writting

$$A_j = K_1 l_j^* + (A_j - K_1 l_j^*),$$

and substituting it into the last equality, after re-arranging the terms, we have

$$-K_1 l_j^* \cdot w_i' = (A_j - K_1 l_j^*) \cdot (r_i^* + w_i').$$

That yields

$$|K_1||l_j^* \cdot w_i'| \le |A_j - K_1 l_j^*|(|r_i^*| + |w_i'|) \le C'\eta(1 + |w_i'|),$$

i.e.,

$$|l_j^* \cdot w_i'| \le \frac{C'\eta(1 + |w_i'|)}{|K_1|}, \quad j \ne i.$$

Besides, w_i' can be expressed in terms of eigenvectors by, observe that $l_i^* \cdot w_i' = 0$,

$$w_i' = \sum_{j \ne i} (l_j^* \cdot w_i') r_j^*$$

Hence, we find

$$|w_i'| \le \sum_{j \ne i} |l_j^* \cdot w_i'||r_j^*| \le \sum_{j \ne i} \frac{C'\eta(1 + |w_i'|)}{|K_1|} = \frac{N-1}{|K_1|} C'\eta(1 + |w_i'|),$$

i.e.,

$$|w_i'| \le \frac{\frac{N-1}{|K_1|} C'\eta}{1 - \frac{N-1}{|K_1|} C'\eta}.$$

Since it is not restrictive to require that

$$C \ge \frac{\frac{N-1}{|K_1|} C'}{1 - \frac{N-1}{|K_1|} C'\eta}.$$

it follows that $\mathrm{Lip}_\epsilon(w_i) \le C\,\eta$, and therefore

$$\begin{aligned} |l_i^* \cdot (\varphi_i(\epsilon, u_0) - u_0)| - \gamma\, |\varphi_i(\epsilon, u_0) - u_0| &= |\epsilon| - \gamma |\epsilon \cdot r_i^* - w_i(\epsilon)| \\ &> |\epsilon| - \gamma(|\epsilon| + \mathrm{Lip}_\epsilon(w_i)|\epsilon|) > |\epsilon| - > \gamma|\epsilon|(1 + C\eta) > 0, \end{aligned}$$

provided γ is chosen such that $\gamma < 1/(1 + C\eta)$, and thus

$$\varphi_i(\epsilon; u_0) \in C_{\gamma,i}.$$

This enables us to replace $\tilde{F}_j$ in (4.23) by F_j. Therefore, the i-Hugoniot curve $\varphi_i(\epsilon; u_0)$ is uniquely defined.

Let us next consider with the relations (4.9). The first equality is obvious. Observe that

$$|\varphi_i'(\epsilon; u_0) - r_i^*| \leq \mathrm{Lip}_\epsilon(w_i) \leq C\,\eta \quad \text{for a.e. } \epsilon \in (-\epsilon_1, \epsilon_1),$$

which implies

$$|\partial\varphi_i(0; u_0) - r_i^*| \leq C\,\eta. \tag{4.24}$$

On the other hand, the upper semi-continuity property of generalized gradients (Proposition 2.2, item c)) shows that given $\epsilon > 0$ there exists $\delta > 0$ such that for all $|u - u_0| < \delta$

$$\partial f([u_0, u]) \subset \partial f(u_0) + \epsilon\,\mathcal{B}(0, 1).$$

The right-hand side of the above inequality being convex we have

$$\mathrm{co}\,\partial f([u_0, u]) \subset \partial f(u_0) + \epsilon\,\mathcal{B}(0, 1).$$

Since the eigenvalues and eigenvectors depend continuously upon their arguments, it follows from the last inclusion that, for any matrix $A(u_0, u) \in \mathrm{co}\,\partial f([u_0, u])$ with i-eigenvalue $\lambda_i(u_0, u)$ and i-eigenvector $r_i(u_0, u)$,

$$\begin{aligned} &\left|\lambda_i(u_0, u) - \lambda_i(u_0)\right| < C''\epsilon, \\ &|r_i(u_0, u) - r_i(u_0)| < C''\epsilon, \end{aligned}$$

for some $C'' > 0$, $\lambda_i(u_0) \in \Lambda_i(u_0)$, and $r_i(u_0) \in R_i(u_0)$. Thus, we get

$$\begin{aligned} &\lambda_i(u_0, \varphi_i(\epsilon; u_0)) \to \lambda_i(u_0), \\ &r_i(u_0, \varphi_i(\epsilon; u_0)) \to r_i(u_0) \quad \text{as } \epsilon \to 0. \end{aligned} \tag{4.25}$$

Combining (4.14), (4.24) and (4.25), we obtain the second and the third inclusions in (4.9).

We are left with checking the shock admissibility inequalities (4.8). As indicated above, we have

$$|\varphi_i'(\epsilon; u_0) - r_i^*| \leq C\eta \quad \text{for a.e. } \epsilon \in (-\epsilon_1, \epsilon_1).$$

Therefore, by our genuine nonlinearity assumption it follows that

$$\begin{aligned} &\sigma_i(\epsilon, u_0) < \sigma_i(0) \in \Lambda_i(u_0) \quad \text{for all } -\epsilon_1 < \epsilon < 0, \\ &\sigma_i(\epsilon, u_0) > \sigma_i(0) \in \Lambda_i(u_0) \quad \text{for all } 0 < \epsilon < \epsilon_1, \end{aligned}$$

so that the first inequality in (4.8) is satisfied and the part $\{\epsilon > 0\}$ of the i-Hugoniot curve is excluded by violating (4.8). Considering the part of the i-Hugoniot curve "between" u_0 and $\varphi_i(\epsilon; u_0)$ as the Hugoniot curve issuing from $\varphi_i(\epsilon; u_0)$,

$$u(s) := \varphi_i(\epsilon; u_0) - (\epsilon - s)\, r_i^* - w_i(\epsilon - s), \quad \epsilon \leq s \leq 0,$$

we find

$$u(0) = u_0, \quad u(\epsilon) = \varphi_i(\epsilon; u_0),$$

and

$$u'(s) = r_i^* + w_i'(\epsilon - s)$$

which satisfies the genuine nonlinearity assumption. The shock speed $\sigma_i(s; \varphi_i(\epsilon; u_0))$ is increasing and, for $-\epsilon_1 < \epsilon < 0$,

$$\sigma_i\big(0; \varphi_i(\epsilon; u_0)\big) > \sigma_i\big(\epsilon; \varphi_i(\epsilon; u_0)\big) \in \Lambda_i\big(\varphi_i(\epsilon; u_0)\big).$$

This establishes the second inequality in (4.8). The proof of Theorem 4.1 is completed. □

For each $i = 1, \ldots, N$ the *i-shock set* $\mathcal{S}_i(u_0)$ is defined to be

$$\mathcal{S}_i(u_0) := \big\{\varphi_i(\epsilon; u_0) \,/\, \epsilon \in (-\epsilon_1, 0]\big\}.$$

Next, we search for self-similar, Lipschitz continuous solutions $u(x, t) = v(\xi)$, $\xi = x/t$ to (1.1) connecting a given left-hand state u_0 to some right-hand state u_1. A rarefaction wave $u(x, t) = v(\xi)$, $\xi = x/t$ satisfies the differential equation

$$-\xi \frac{dv}{d\xi}(\xi) + \frac{d}{d\xi} f(v(\xi)) = \big(-\xi\, I + Df(v(\xi))\big) \frac{dv}{d\xi}(\xi) = 0. \tag{4.26}$$

If (4.26) holds in the usual sense, then there exist right-eigenvector $r_i(v(\xi))$ and eigenvalues $\lambda_i(v(\xi))$ of $Df(v(\xi))$, and a scalar function $c(\xi)$ such that for all relevant values ξ:

$$\begin{aligned} &\frac{dv}{d\xi}(\xi) = c(\xi)\, r_i(v(\xi)), \\ &\xi = \lambda_i(v(\xi)). \end{aligned} \tag{4.27}$$

The function $\xi \mapsto r_i(v(\xi))$ is L^∞ and continuous almost everywhere. Since the right-hand side of (4.27) may be discontinuous, we have to understand solutions of (4.27) in the sense of Filippov [4] and Dafermos [2].

Let us consider the following ordinary differential problem

$$\begin{aligned} &\frac{d\tilde{v}}{ds}(s; u_0) = r_i(\tilde{v}(s; u_0)), \quad \text{a.e. } s \in [0, \epsilon_1), \\ &\tilde{v}(0; u_0) = u_0. \end{aligned} \tag{4.28}$$

For ϵ_1 sufficiently small, a solution of (4.28) in the sense of Filippov exists (see [4]). Precisely, there exists a Lipschitz continuous mapping $\tilde{v}(s; u_0)$, $s \in [0, \epsilon_1)$ satisfying

$$\begin{aligned} &\frac{d\tilde{v}}{ds}(s; u_0) \in \bigcap_{\delta>0} \overline{co}\, r_i\Big(\tilde{v}(s; u_0) + \delta\mathcal{B}(0, 1)\Big) \quad \text{a.e. in } [0, \epsilon_1), \\ &\tilde{v}(0; u_0) = u_0. \end{aligned}$$

The fact that r_i is continuous almost everywhere along the curve $\tilde{v}(.; u_0)$ yields

$$\bigcap_{\delta>0} \overline{co}\, r_i\Big(\tilde{v}(s; u_0) + \delta\mathcal{B}(0, 1)\Big) = \Big\{r_i(\tilde{v}(s; u_0))\Big\} \quad \text{a.e. in } [0, \epsilon_1).$$

The last equality simply means that the function $\tilde{v}(.; u_0)$ is a solution of (4.28) in the usual sense as well. Thanks to the assumption of genuine nonlinearity, the function $\lambda_i(\tilde{v}(s; u_0))$ is strictly increasing and admits a Lipschitz continuous inverse, denoted by

$$\begin{aligned} \psi : [\lambda(u_0), \lambda(\tilde{v}(\epsilon_1; u_0))] &\to [0, \epsilon_1] \\ \xi &\mapsto s = \psi(\xi), \end{aligned}$$

which is increasing as well. We now claim that the function

$$v(\xi) := \tilde{v}(\psi(\xi); u_0), \quad \xi \in J := [\lambda(u_0), \lambda(\tilde{v}(\epsilon_1; u_0))],$$

is a solution of (4.24). Clearly, v is Lipschitz continuous. Besides, let $\Omega_{\tilde{v}}$ be the set of all points at which $\tilde{v}$ fails to be differentiable, which has Lebesgue measure zero. Set

$$E = \{\xi \in J : \psi(\xi) \in \Omega_{\tilde{v}}\}.$$

By [3, Th. A.1] the measure $D\psi$ vanishes on E:

$$|D\psi|(E) = 0. \tag{4.29}$$

Therefore, (4.26) holds in the set E. For $\xi \in J \setminus E$ the function v satisfies

$$\begin{aligned} v'(\xi) &= \frac{d}{ds}\tilde{v}(\psi(\xi))\frac{d}{d\xi}\psi(\xi) \\ &= r_i(\tilde{v}(\psi(\xi)))\frac{d}{d\xi}\psi(\xi) = \frac{d}{d\xi}\psi(\xi)r_i(v(\xi)). \end{aligned}$$

From the above analysis we obtain the wave curve

$$\epsilon \mapsto \phi_i(\epsilon; u_0) := \tilde{v}(\epsilon; u_0)$$

and arrive at the following conclusion.

Theorem 4.2. *Given $u_0 \in \mathcal{B}(u_*, \delta_0)$ and $i = 1, \ldots, N$, there exists a Lipschitz continuous curve $[0, \epsilon_1) \ni \epsilon \mapsto \phi_i(\epsilon; u_0) \in \mathcal{B}(u_*, \delta_0)$ (defined over some small interval $[0, \epsilon_1)$) such that the state u_0 can be connected to $\phi_i(\epsilon; u_0)$ from the right by a rarefaction wave.*

We define the *i-rarefaction curve* $\mathcal{R}_i(u_0)$ by

$$\mathcal{R}_i(u_0) := \big\{\phi_i(\epsilon; u_0) \,/\, \epsilon \in [0, \epsilon_1)\big\}.$$

The *i-wave curve* issuing from u_0 is

$$\mathcal{W}_i(u_0) := \mathcal{S}_i(u_0) \cup \mathcal{R}_i(u_0).$$

We are at the position to state the main result of this section.

Theorem 4.3. *There exist $\delta_1 > 0$ and $\epsilon_1 > 0$ such that for every $u_0 \in \mathcal{B}(u_*, \delta_1)$ and $i = 1, \ldots, N$, there is a wave curve issuing from u_0*

$$\mathcal{W}_i(u_0) := \{\psi_i(\epsilon^i; u_0) \,/\, \epsilon^i \in (-\epsilon_1, \epsilon_1)\}.$$

Given data $u_l, u_r \in \mathcal{B}(u_*, \delta_1)$, the corresponding Riemann problem (1.1)-(1.2) *admits a self-similar, piecewise Lipschitz continuous solution made of $N + 1$ constant states*

$$u_l = u_0, u_1, \ldots, u_N = u_r,$$

separated by elementary waves. The intermediate states satisfy $u_j \in \mathcal{W}_j(u_{j-1})$ with $u_j = \psi_j(\epsilon^j, u_{j-1}) := \psi_j(\epsilon^j)(u_{j-1})$ for some (wave strength) $\epsilon^j \in (-\epsilon_1, \epsilon_1)$. The states u_{j-1} and u_j are connected by either a rarefaction wave if $\epsilon^j \geq 0$ or by a shock satisfying the generalized Lax shock inequalites (4.8) *if $\epsilon^j < 0$.*

Proof. Consider the mapping obtained by combining wave curves together

$$\epsilon \doteq (\epsilon^1, \epsilon^2, \dots, \epsilon^N) \mapsto \Psi(\epsilon) = \psi_N(\epsilon^N) \circ \psi_{N-1}(\epsilon^{N-1}) \circ \dots \circ \psi_1(\epsilon^1)(u_l) - u_l.$$

It satisfies

$$\Psi(0) = 0.$$

According to Theorems 4.1 and 4.2 we have

$$\psi_i(\epsilon^i)(u) \in u + \epsilon^i\, R_i(u) + o(\epsilon^i)\mathcal{B}(0,1).$$

Hence, we get

$$\Psi(\epsilon) \subset \sum_i \epsilon^i\, R_i(v_i) + o(\epsilon)\mathcal{B}(0,1),$$

where

$$\begin{aligned} v_i &= \psi_{i-1}(\epsilon^{i-1}) \circ \dots \circ \psi_1(\epsilon^1)(u_l), \quad \text{for } i = 2, \dots, N, \\ v_1 &= u_l. \end{aligned}$$

Thus, we have

$$\partial\Psi(0) \subset \big(R_1(u_l), R_2(v_2), \dots, R_N(v_N)\big). \tag{4.30}$$

The upper semi-continuity of the generalized gradient,

$$\partial f(v_i) \subset \partial f(u_l) + \epsilon'\mathcal{B}(0,1) \quad \text{for } v_i \text{ near } u_l,$$

implies that R_i depends continuously on its argument upon small perturbation, i.e.,

$$R_i(v_i) \subset R_i(u_l) + O(\epsilon')\mathcal{B}(0,1).$$

We can assume that η and ϵ' are sufficiently small so that the last estimate and the hyperbolicity property imply that any selection of the vector sets $R_i(v_i)$ is a basis of $\mathbb{R}^N$. Therefore, the matrix $\partial\Psi(0)$ shown by (4.30) is of maximal rank. Applying the inverse function theorem (Theorem 2.5) we conclude that, for $|u_r - u_l|$ sufficiently small, there exists a unique vector $\epsilon_0 = (\epsilon_0^1, \epsilon_0^2, \dots, \epsilon_0^N)$ such that

$$\Psi(\epsilon_0) = u_r - u_l.$$

In other words, we have

$$\psi_N(\epsilon_0^N) \circ \psi_{N-1}(\epsilon_0^{N-1}) \circ \cdots \circ \psi_1(\epsilon_0^1)u_l = u_r,$$

which completes the proof of Theorem 4.3. □

5 A model from compressible fluid dynamics

In this last section we consider the Riemann problem for the so-called p-system

$$\begin{aligned} u_t + p(v)_x &= 0, \\ v_t - u_x &= 0. \end{aligned} \tag{5.1}$$

Here $v > 0$ and u denote the specific volume and the velocity of the fluid, respectively. The pressure $p = p(v)$ is assumed to be smooth everywhere in $v > 0$ (say of class C^2) *except* at one point v_*. More precisely, we assume that

$$\begin{aligned} & p'_-(v_*) < p'_+(v_*), \quad p''(v_*\pm) > 0, \\ & p'(v) < 0, \quad p''(v) > 0 \quad \text{for } v \neq v_*, \\ & \lim_{v\to 0+} p(v) = +\infty, \quad \lim_{v\to +\infty} p(v) = 0. \end{aligned} \tag{5.2}$$

These conditions are typical in models arising in fluid dynamics when the equation of state is defined by distinct formulas above and below some critical threshold. We set $U = (v, u)^T$ and $f(U) = (-u, p(v))^T$, so that (5.1) has the form (1.1) with U playing the role of u in (1.1). For $v \neq v_*$, the Jacobian matrix of the system is

$$Df(U) = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix} \tag{5.3a}$$

and the generalized Jacobian (in the sense of Section 2) at the point (v_*, u) is

$$\partial f(v_*, u) = \begin{pmatrix} 0 & -1 \\ [p'_-(v_*), p'_+(v_*)] & 0 \end{pmatrix}. \tag{5.3b}$$

Eigenvalues and eigenvectors are given by

$$\begin{aligned} & \lambda_1(v) \in \{-\sqrt{\bar\lambda} \,/\, \bar\lambda \in \partial p(v)\}, \qquad \lambda_2(v) \in \{\sqrt{\bar\lambda} \,/\, \bar\lambda \in \partial p(v)\}, \\ & r_1(v) = (1, -\lambda_1(v))^T, \qquad r_2(v) = (-1, \lambda_2(v))^T. \end{aligned} \tag{5.4}$$

The system (5.1) is strictly hyperbolic since

$$\lambda_1(v) < 0 < \lambda_2(v).$$

Furthermore, away from $v \neq v_*$ both characteristic fields of the system are genuinely nonlinear since

$$\nabla\lambda_i(v) \cdot r_i(v) = \frac{p''(v)}{2\sqrt{-p'(v)}} > 0.$$

Finally, we set also

$$\begin{aligned}&\Omega_- := \{(v,u)\,/\,0 < v < v_*\}, \quad \Omega_+ := \{(v,u)\,/\,v > v_*\},\\&\Omega_* := \{(v,u)\,/\,v = v_*\}.\end{aligned} \tag{5.5}$$

The first is decreasing while the second is increasing. We determine the rarefaction waves for the system (5.1) as follows. Let $U_0 = (v_0, u_0)$ be a fixed state. The rarefaction waves issued from U_0 are continuous solutions $U(\xi) = (v(\xi), u(\xi))$ (in each interval where $u(\xi) \notin \Omega_*$) to the problem

$$\begin{aligned}&\frac{d}{d\xi}U(\xi) = \alpha(\xi)\,r_i(v(\xi)), \quad \xi \ge \xi_0,\\&\xi = \lambda_i(v(\xi)), \quad U(\xi_0) = U_0,\end{aligned} \tag{5.6}$$

where $i = 1$ or 2 and $\alpha = \alpha(\xi)$ is some real-valued function. Differentiating the relation $\xi = \lambda_i(v(\xi))$ away from the region Ω_* yields

$$\begin{aligned}1 &= \nabla\lambda_i(v(\xi)) \cdot \frac{dv}{dv}(\xi)\\&= \alpha(\xi)\,\nabla\lambda_i(v(\xi)) \cdot r_i(v(\xi)).\end{aligned} \tag{5.7}$$

Substituting (5.7) into (5.6) we obtain

$$v'(\xi) = (-1)^{i+1}\frac{2\sqrt{-p'(v)}}{p''(v)}, \qquad u'(\xi) = \frac{2 - p'(v)}{p''(v)}.$$

Since $v'(\xi) \neq 0$ this system of ODE's enables us to write $u = u(v; U_0)$

$$\frac{du}{dv} = (-1)^{i+1}\sqrt{-p'(v)}. \tag{5.8}$$

For $i = 1$ the condition $\lambda_1(v) > \lambda_1(v_0)$ yields $p'(v) > p'(v_0)$ and, therefore, $v > v_0$, since p' is strictly increasing by assumption. Hence, from (5.8) it follows that the 1-rarefaction curve is

$$\mathcal{R}_1(U_0) = \left\{u(v; U_0) = u_0 + \int_{v_0}^{v} \sqrt{-p'(y)}\,dy, \quad v > v_0\right\}. \tag{5.9}$$

Similarly, for $i = 2$ the 2-rarefaction curve is

$$\mathcal{R}_2(U_0) = \left\{u(v; U_0) = u_0 - \int_{v_0}^{v} \sqrt{-p'(y)}\,dy, \quad v < v_0\right\}. \tag{5.10}$$

For $U_1 \in \mathcal{R}_i(U_0)$ the i-rarefaction wave $\xi \mapsto U(\xi)$ connecting U_0 to U_1 on the right is given by

$$U(\xi) = \begin{cases} U_0, & \xi \le \lambda_i(v_0), \\ \big(v(\xi), u(v(\xi); u_0)\big), & \lambda_i(v_0) \le \xi \le \lambda_i(v_1), \\ u_1, & \xi \ge \lambda_i(v_1). \end{cases} \tag{5.11}$$

It is solely a Lipschitz continuous function in the variable $\xi = x/t$. There may exist a new intermediate constant state, which is a direct consequence of the discontinuity in characteristic speed. The profile $v(\xi)$ in (5.11) is determined by inverting the relation $\xi = \lambda_i(v(\xi))$. For $i = 1$ one gets

$$v(\xi) = \begin{cases} (-p')^{-1}(\xi^2), & \xi < -\sqrt{-p'_-(v_*)} \quad \text{or} \\ & -\sqrt{-p'_+(v_*)} < \xi < -\sqrt{-p'(+\infty)} \\ v_*, & -\sqrt{-p'_-(v_*)} \le \xi \le -\sqrt{-p'_+(v_*)}, \end{cases} \tag{5.12}$$

and, for $i = 2$,

$$v(\xi) = \begin{cases} (-p')^{-1}(\xi^2), & \sqrt{-p'(+\infty)} < \xi < \sqrt{-p'_+(v_*)} \quad \text{or} \\ & \xi > \sqrt{-p'_-(v_*)} \\ v_*, & -\sqrt{-p'_+(v_*)} \le \xi \le \sqrt{-p'_-(v_*)}. \end{cases} \tag{5.13}$$

We now summarize the above discussion.

Proposition 5.1. *For each $U_0 = (v_0, u_0)$ such that $v_0 > 0$ and for each $i = 1, 2$ the rarefaction curve $v \mapsto u = u(v; U_0)$ issued from U_0, $\mathcal{R}_i(U_0)$, is globally defined by* (5.9) *and* (5.10). *For $i = 1$ this mapping is increasing and concave in v and for $i = 2$ it is decreasing and convex. Moreover, each mapping $u(v; U_0)$ is locally Lipschitz continuous in $(v; U_0)$. For each fixed U_0 it is of class C^2 in the variable $v \ne v_*$, but its derivative exhibits a jump at $v = v_*$. The same regularity holds true for $u(v; U_0)$ considered as a function of v_0 while keeping v and u_0 fixed.*

We turn to the investigation of shock waves of the system (5.1). That is, discontinuous solutions of (1.1) connecting two constant states $U_0 = (v_0, u_0)$ and $U = (v, u)$ at some speed s. Using the Rankine-Hugoniot condition and the generalized Lax shock inequalities ($i = 1, 2$)

$$\lambda_{i+}(v) < s < \lambda_{i-}(v_0), \tag{5.14}$$

and relying on the assumptions (5.2) and (5.5) we easily determine the shock curves:

$$\mathcal{S}_1(U_0) := \left\{ u(v; U_0) = u_0 - \sqrt{-\Big(p(v) - p(v_0)\Big)\,(v - v_0)}, \quad 0 < v < v_0 \right\},$$

$$s = s_1(v; v_0) = -\sqrt{-\frac{p(v) - p(v_0)}{v - v_0}}, \tag{5.15}$$

and

$$\mathcal{S}_2(U_0) := \left\{ u(v; U_0) = u_0 + \sqrt{-\Big(p(v) - p(v_0)\Big)\,(v - v_0)}, \quad v > v_0 \right\},$$

$$s = s_2(v; v_0) := \sqrt{-\frac{p(v) - p(v_0)}{v - v_0}}. \tag{5.16}$$

We conclude that:

Proposition 5.2. *For each $U_0 = (v_0, u_0)$ (with $v_0 > 0$) and each $i = 1, 2$ the shock curve $v \mapsto u(v; U_0)$ issued from U_0, $\mathcal{S}_i(U_0)$, is globally defined by* (5.15) *and* (5.16)*. For $i = 1$ the mapping $u(v; U_0)$ is increasing and concave in the v variable and, for $i = 2$, is decreasing and convex. Moreover, each mapping $u(v; U_0)$ is locally Lipschitz continuous in $(v; U_0)$. For U_0 fixed it is of class C^2 in the variable $v \neq v_*$ but its derivative exhibits a jump at $v = v_*$. The shock speed is a locally Lipschitz continuous function, which is of class C^2 at $v \neq v_*$. Finally, we have*

$$u(v_0; U_0) = u_0, \quad u'(v_0; U_0) = (-1)^{i+1}\sqrt{-p'_{\mp}(v_0)},$$
$$s_i(v_0; v_0) = (-1)^i\sqrt{-p'_{\mp}(v_0)}.$$

If, in addition to the assumption (5.2), the function p satisfies (for instance) $\int_1^\infty \sqrt{-p'(v)}dv = +\infty$, then the Riemann problem for the p-system admits a unique self-similar solution made of shock and rarefaction waves.

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